Non-constructivity of the Cantor-Bernstein theorem

P. Pradic
j.w.w. C. E. Brown

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Constructivity (1/2)

Theorem

$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

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Morality

$\rightsquigarrow$ Not all mathematical arguments are equally informative.

Constructivity (2/2)

In broad strokes

Reject excluded middle and reductio ad absurdum.

$$A \vee \neg A \qquad\qquad \neg\neg A \Rightarrow A$$

- Interesting for a variety of reasons, non-philosophical or otherwise
- Large amounts of mathematics can still be formalized

abstract nonsense, finitary combinatorics, $(\mathbb{Q}, <)$

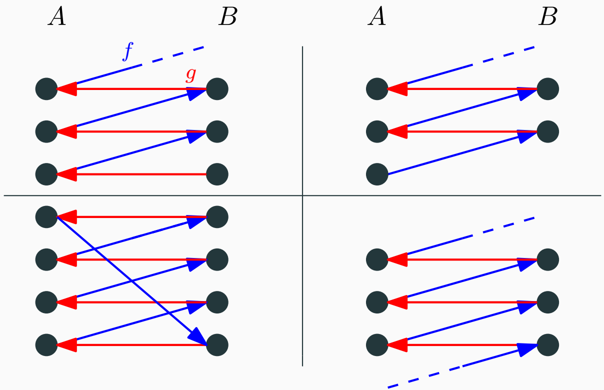
Some things that break down easily

- decidability of equality for $\mathbb{R}$ or $2^{\mathbb{N}}$ $\qquad \forall x, y \in 2^{\mathbb{N}}.\ x = y \vee x \neq y$
- infinitary combinatorics
- ordinal theory

- Some taboos: $\mathbb{R}_{\text{Cauchy}} \cong \mathbb{R}_{\text{Dedekind}}$ (as fields), $2^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$ (as sets)

The CB theorem

If there exists injection $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \cong B$.



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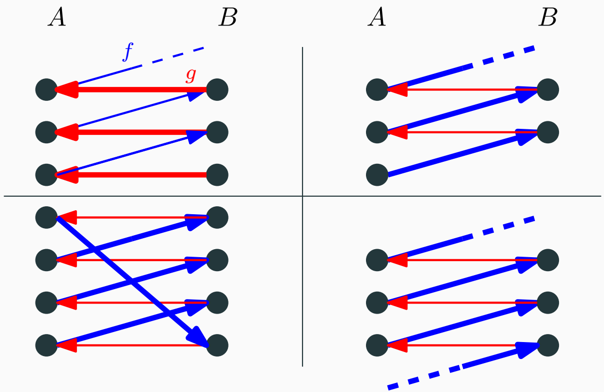

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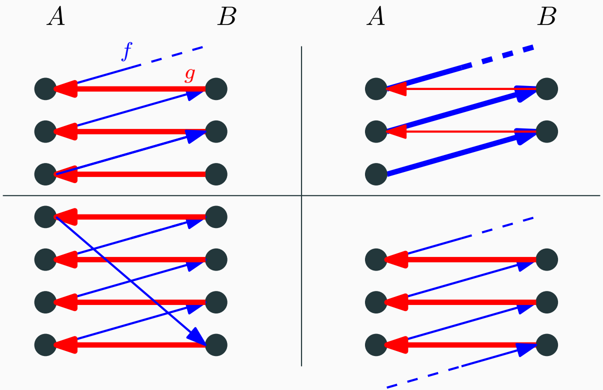

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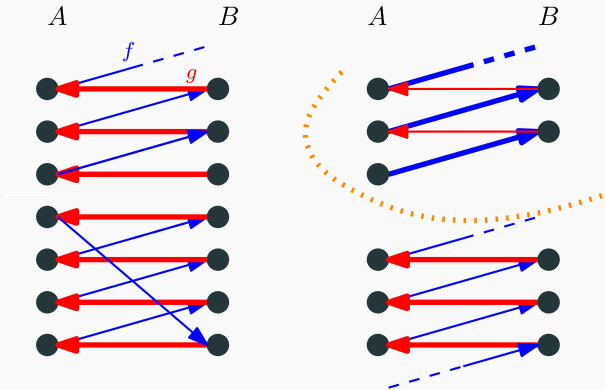
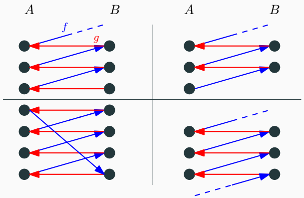


$\longrightarrow$ excluded middle used to show that we have a partition

What (can't) we do constructively?

- We can ask for the successor of a node in $f \cup g^{-1}$...
- ...but not predecessor

Taboo: "am I in the range of f?"



Even if we could, that would not be enough!

Taboo: "do I have finitely many predecessors?"

Folklore

Cantor-Bernstein fails for models of intuitionistic set theory.

- For the gros topos, $2^{\mathbb{N}} \ncong \mathbb{N}^{\mathbb{N}}$

  $\mathbb{N}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$ constructively as usual
- In Kleene realizability, easy recursion-theoretic counterexamples.

  e.g. $\mathbb{N}$ vs $\mathbb{N} + \text{Halt}$

As non-constructive as can be

Theorem

Over intuitionistic set theory (IZF), the Cantor-Bernstein theorem implies excluded middle.

Plan:

- Proof of a slightly weaker statement (due to Banaschewski and Brümmer)
- Introduce $\mathbb{N}_\infty$ and its effective searchability (due to Escardó)
- Conclude

Quick preliminaries

Remark **(b/c separation axiom)**

Let $\bullet$ be such that $\bullet \notin \mathbb{N}, 2^{\mathbb{N}}$. Then excluded middle is equivalent to

$$\forall A \subseteq \{\bullet\}.\ A = \emptyset \vee \exists x \in A$$

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- 2 is the two-element set
- *cannot* be identified with truth-values/$\mathcal{P}(\{\bullet\})$
- we will mostly play around with a singleton set $\{\bullet\}$, $\mathbb{N}$ and $2^{\mathbb{N}}$.

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For the sequel

Assume $\bullet \notin \mathbb{N} \cup 2^{\mathbb{N}}$ to be distinguishable from elements of $\mathbb{N}$ and $2^{\mathbb{N}}$

$$\forall x \in \{\bullet\} \cup \mathbb{N} \cup 2^{\mathbb{N}}.\ x \in \mathbb{N} \vee x \in 2^{\mathbb{N}} \vee x = \bullet$$

Banaschewski and Brümmer's reversal

A strengthening of Cantor-Bernstein (CBBB)

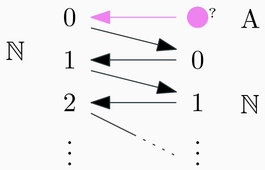
If there exists injection $f : A \to B$ and $g : B \to A$, then there exists $h : A \cong B$ *with* $h \subseteq f \cup g^{-1}$

Theorem (Banaschewski and Brümmer 1986)

Over IZF, CBBB implies excluded middle.

Fix $A \subseteq \{\bullet\}$ and build maps $f : \mathbb{N} \to A \cup \mathbb{N}$ and $g : A \cup \mathbb{N} \to \mathbb{N}$

$$f(n) := n \qquad g(\bullet) := 0 \qquad g(n) := n + 1$$



Is A inhabited or not?
$\to$ is $h(0) = \bullet$ or 0?

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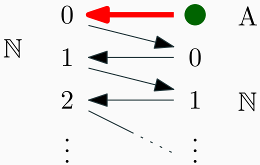

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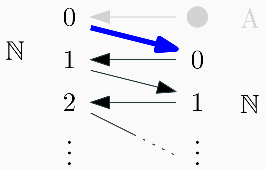

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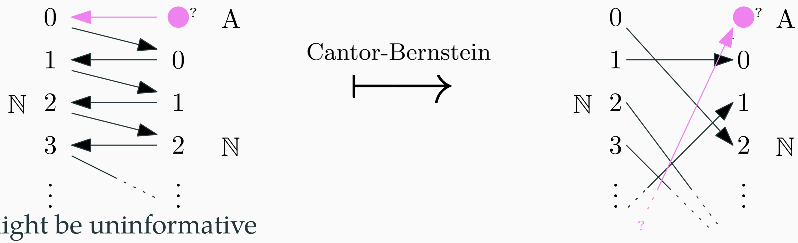
No!

For general Cantor-Bernstein



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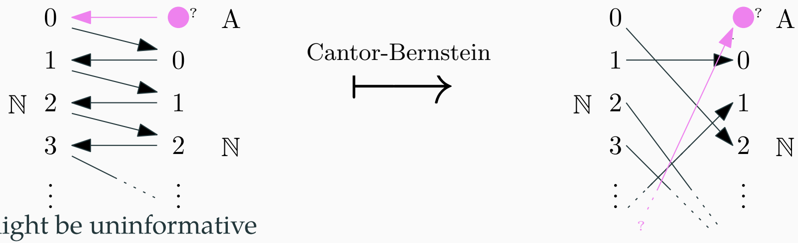

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Idea

Replace $\mathbb{N}$ with another domain $\mathbb{N}_\infty$ for which we can ask our question

"For any $h : \mathbb{N}_\infty \to A \cup \mathbb{N}_\infty$, is $\bullet \in h(\mathbb{N}_\infty)$?"

Meet $\mathbb{N}_\infty$

Definition

$$\mathbb{N}_\infty \quad := \quad \{p \in 2^{\mathbb{N}} \mid \exists^{\leq 1} n \in \mathbb{N}.\ p(n) = 1\}$$

- Alternative definition: final coalgebra for $X \mapsto 1 + X$ — streams of • that might halt
- Call ∞ the sequence $n \mapsto 0$ — the infinite stream
- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let's write it $n \mapsto \underline{n}$.

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- Embedding $\mathbb{N} \to \mathbb{N}_\infty$: let's write it $n \mapsto \underline{n}$.
- **Classically,** $\mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\infty\}$ — equivalent to Σ^0_1-excluded middle
- Can constructively define addition, but not subtraction or an equality map $\mathbb{N}^2_\infty \to 2$

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Theorem (Escardó 2013)

There is a map $\varepsilon : 2^{\mathbb{N}_\infty} \to \mathbb{N}_\infty$ that picks witnesses

$$\forall p \in 2^{\mathbb{N}_\infty}.\;\; (\exists n \in \mathbb{N}_\infty.\, p(n) = 1) \implies p(\varepsilon(p)) = 1$$

provably in constructive set theory

(nice to compare and contrast with $2^{\mathbb{N}}$...)

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$$\varepsilon(p) = \begin{cases} \underline{0} & \text{if } p(\underline{0}) = 1 \\ \underline{\mathsf{Succ}}(\varepsilon(p \circ \underline{\mathsf{Succ}})) & \text{otherwise} \end{cases}$$

where
$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{n \mapsto n+1} & \mathbb{N} \\ \downarrow & & \downarrow \\ \mathbb{N}_\infty & \xrightarrow[\underline{\mathsf{Succ}}]{} & \mathbb{N}_\infty \end{array}$$

Recursive version in Haskell

```haskell
type Ninfty = Int -> Bool

ofInt :: Int -> Ninfty
ofInt n i = n == i

epsilon :: (Ninfty -> Bool) -> Ninfty
epsilon p k = not exSmallerWitness && p (ofInt k)
  where exSmallerWitness = any (p . ofInt) [0..k-1]
```

Proof

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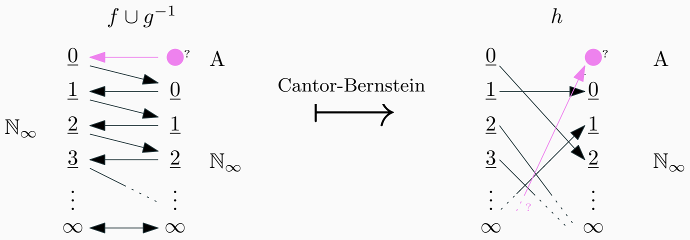

- Define $p \in 2^{\mathbb{N}_\infty}$ by $p(n) :=$ "$h(n) = \bullet$"
- Conclude using $p(\varepsilon(p)) = 1 \iff \bullet \in A$

Further thoughts

Remarks

- Trick very much unlike the folklore examples

$\rightarrow$ does not give concrete counterexamples in 2-valued models

- Requires the axiom of infinity — consider $\mathcal{C}^{op} \rightarrow \mathsf{Finset}$ for finite $\mathcal{C}$

Extensions?

- Restriction to e.g., sets with discrete equalities?
- Any relation to investigations of the CB property in more general categories?

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Thanks for listening! Questions?