

Some proof-theoretical approaches to Monadic Second-Order logic

PhD defense

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Goal : check safety of engineered systems



- ▶ “The green and red lights are not on at the same time”
- ▶ “Orange is flashed before red”
- ▶ ...

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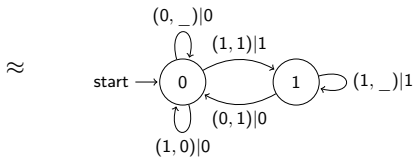


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Some more complicated devices:



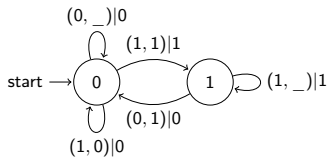
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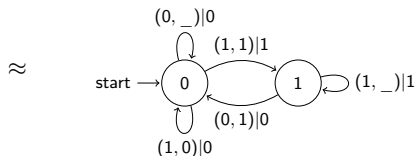


\approx



Typical task: Given a specification φ (logical formula)...

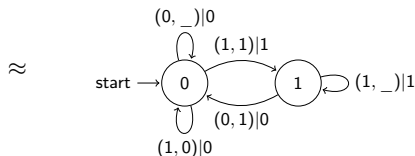
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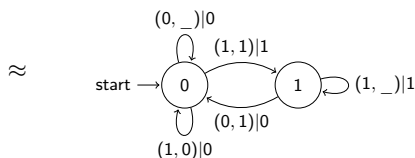
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Answer whether yes or no a system satisfies φ ?

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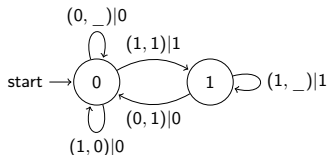
Synthesis

Generate a system satisfying φ from scratch.

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Generate a system satisfying φ from scratch.

Decide logic?

Hilbert's dream

The more extreme version of Hilbert's program (1920s):

- ▶ Reduce mathematics to formalized arithmetics.
- ▶ A mechanical method to decide the (mathematical) **truth**.

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Decidable subcases

- ▶ Logics over fixed finite domains.
- ▶ Monadic Second Order (MSO) logic over infinite words.

Proof theory at a very high level

Formalize mathematically what is a correct proof.

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$\pi + e$ is transcendental or $e \cdot \pi$ is transcendental (or both are).

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A *constructive proof* would be more informative.

proofs \longrightarrow computable witnesses

Monadic Second-Order logic (MSO)

- ▶ A fragment of Second-Order logic.
- ▶ Algorithmically decidable over

\mathbb{N}, \mathbb{Q} , the infinite binary tree $\{0, 1\}^*$, ...

- ▶ Subsumes many verification logics.

LTL, CTL, ...

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Decidable \neq constructive

Soundness of decision procedures \Leftarrow non-constructive theorems.

- ▶ Over \mathbb{N} : infinite Ramsey theorem, weak König's Lemma.
- ▶ Over $\{0, 1\}^*$: determinacy of infinite parity games.

How (non)-constructive is MSO?

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What axiomatic strength characterizes a given MSO theory?

- ▶ With H. Michalewski, L. Kołodziejczyk and M. Skrzypczak in Warsaw.

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- ↪ Metatheoretical analysis of Büchi's decidability theorem.

When can we extract computational content from MSO proofs?

- ▶ With C. Riba in Lyon.
- ↪ Refinement of $\text{MSO}(\mathbb{N})$ with witness extraction.

Monadic Second-Order logic

Part I: Reverse Mathematics

Part II: proof systems for Church's synthesis

Conclusion

Syntax of MSO(\mathbb{N})

$$\varphi, \psi ::= n \in X \mid n < k \mid \exists n \varphi \mid \exists X \varphi \mid \neg \varphi \mid \varphi \wedge \psi$$

- ▶ Can be regarded as a subsystem of Second-Order Arithmetic
- ▶ Standard model: $n \in \mathbb{N}$, $X \in \mathcal{P}(\mathbb{N})$
- ▶ Only *unary* predicates.

no pairing, no addition

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Typical $\text{MSO}(\mathbb{N})$ -definable properties

- ▶ “The set $X \subseteq \mathbb{N}$ is infinite.”
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Typical MSO(\mathbb{N})-definable properties

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Corresponds exactly to sets recognizable by automata over infinite words.

- ▶ Infinite words: regard sets as sequences of bits through $\mathcal{P}(\mathbb{N}) \simeq 2^\omega$
- ▶ $\varphi(X_1, \dots, X_k)$: formula over Σ^ω for $\Sigma = 2^k$

Definition

A non-deterministic Büchi automaton (NBA) $\mathcal{A} : \Sigma$ is a tuple (Q, q_0, δ, F)

- ▶ Q is a finite set of states, $q_0 \in Q$
- ▶ transition function $\delta : \Sigma \times Q \rightarrow \mathcal{P}(Q)$
- ▶ $F \subseteq Q$ *accepting states*

Recognizes languages of infinite words $\mathcal{L}(\mathcal{A}) \subseteq \Sigma^\omega$:

$w \in \mathcal{L}(\mathcal{A})$ iff there is a run over $w \in \Sigma^\omega$ hitting F infinitely often

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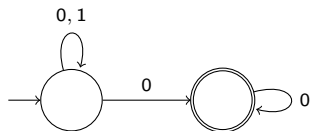
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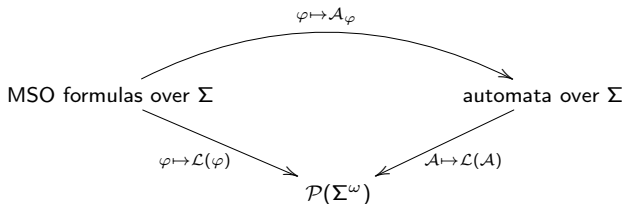
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Example:



$\mathcal{L}(\mathcal{A}) =$ streams with finitely many 1.



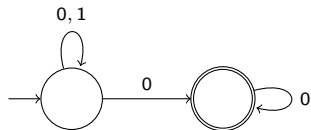
Decidability [Büchi (1962)]

MSO over infinite words is decidable.

- ▶ Proof idea: automata theoretic-construction for each logical connective.
- ▶ Hard case for infinite words: negation \neg .

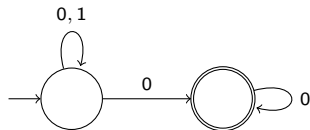
corresponds to *complementation*

For finite word automata: easy complementation for *deterministic* automata.



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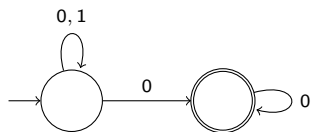
Theorem [McNaughton (1968)]

Non-deterministic Büchi automata can be determinized into *Rabin automata*.

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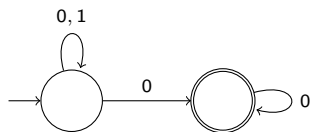
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Quantify how non-constructive they are?

Monadic Second-Order logic

Part I: Reverse Mathematics

Reverse Mathematics

Büchi's theorem

Beyond infinite words

Part II: proof systems for Church's synthesis

Conclusion

- ▶ A framework to analyze axiomatic strength.
- ▶ Vast program.

[Friedman, Simpson, Steele 70s]

Methodology

- ▶ Consider a theorem T formulated in second-order arithmetic.
- ▶ Work in the weak theory RCA_0 .
- ▶ Target some natural axiom A such that $\text{RCA}_0 \not\vdash A$.
- ▶ Show that $\text{RCA}_0 \vdash A \Leftrightarrow T$.

Essentially independence proofs. . .

- ▶ Similar in spirit to statements like

“Tychonoff’s theorem is equivalent to the axiom of choice.”

The big five

Π_1^1 Comprehension

$\Pi_1^1\text{-CA}_0$

\iff

Lusin's separation theorem

\Downarrow

Transfinite Recursion

ATR_0

\iff

Determinacy of open games

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Recursive Comprehension

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Outliers: infinite Ramsey for pairs, determinacy statements.

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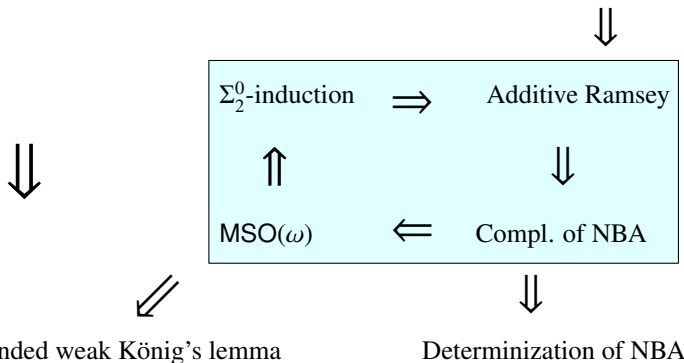
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\rightsquigarrow Where does Büchi's theorem sit in this hierarchy?

Weak König's lemma

Infinite Ramsey theorem



The Logical Strength of Büchi's Decidability Theorem

[Kołodziejczyk, Michalewski, P., Skrzypczak, 2016]

Theorem [Kołodziejczyk, Michalewski (2015)]

Decidability of MSO over the infinite binary tree is not provable in $\Pi_2^1\text{-CA}_0$.

- ▶ Rabin's theorem requires much higher axiomatic strength.
 - ▶ Roughly on par with determinacy of infinite parity games.

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- ▶ Intermediate cases?

MSO over the rationals ($\text{MSO}(\mathbb{Q})$)

- ▶ Decidable via a reduction to the infinite tree.
- ▶ Cover all countable linear orders.
- ▶ Direct algebraic decidability proofs.

[Shelah (1975)], [Carton, Colcombet, Puppis (2013)]

Strength of additive Ramsey over \mathbb{Q} and $\text{MSO}(\mathbb{Q})$

Theorem

[Kołodziejczyk, Michalewski, P., Skrzypczak]

Over RCA_0 , the following are equivalent:

- ▶ the shuffle principle [Carton, Colcombet, Puppis (2013)]
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The classical theory $\text{MSO}(\mathbb{Q})$ has a sentence equivalent to $\Pi_1^1\text{-CA}_0$.

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Conjecture

Over RCA_0 , the following are equivalent:

- ▶ The axiom of finite Π_1^1 -recursion.
- ▶ Determinacy of infinite weak parity games. $BC(\Sigma_1^0)$ games
- ▶ Soundness of the decision algorithm for $\text{MSO}(\mathbb{Q})$.

Monadic Second-Order logic

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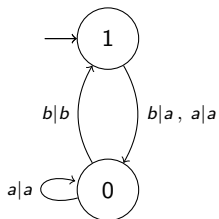
Church's synthesis and witness extraction

Constructive proof systems

Categorical/syntactic approach

Conclusion

Church's synthesis (1/2): causal functions



Causal/synchronous stream functions $f : \Sigma^\omega \rightarrow \Gamma^\omega$

- ▶ Interpret $n \in \mathbb{N}$ as **time steps**.
- ▶ Lifted from functions $\hat{f} : \Sigma^+ \rightarrow \Gamma$ as

$$\begin{aligned} \hat{f} : \Sigma^\omega &\rightarrow \Gamma^\omega \\ s &\mapsto n \mapsto f(s(0) \dots s(n)) \end{aligned}$$

i.e., the output does not depend on the future.

- ▶ Focus on *finite-state* causal functions.
(Correspond to *Mealy machines*)

- ▶ All f.s. causal functions are recursive.
- ▶ All causal functions are continuous.
- ▶ Some recursive functions are not causal.

$$w \mapsto n \mapsto w_{n+1}$$

Church's synthesis problem

Given a formula $\varphi(X, Y)$, find a f. s. causal $f : \Sigma^\omega \rightarrow \Gamma^\omega$ such that

$$\forall w \varphi(w, f(w))$$

Church's synthesis (2/2): the Büchi-Landweber theorem

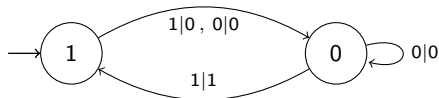
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Example (inspired from [Thomas (2008)]):

- ▶ $\varphi(X, Y) \equiv (X \text{ infinite} \Rightarrow Y \text{ infinite})$ and $\forall i (i \in Y \Rightarrow i + 1 \notin Y)$



Theorem [Büchi-Landweber (1969)]

Algorithmic solution for $\varphi(X, Y)$ in MSO.

- ▶ Algorithmically costly...

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Church's synthesis reminiscent of extraction from proofs:

$$\text{MSO} \vdash \forall x \exists y \varphi(x, y) \quad \stackrel{?}{\implies} \quad \exists f \text{ f.s. causal } \forall x \varphi(x, f(x))$$

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Classical theorems in MSO

- ▶ Excluded middle (subtle point $\{0, 1\}^\omega$ vs $\mathcal{P}(\mathbb{N})$)
- ▶ The infinite pigeonhole principle
- ▶ Instances of additive Ramsey

\rightsquigarrow No algorithmic witnesses for $\forall \exists$ theorems.

Goal: a refinement of $\text{MSO}(\mathbb{N})$ with extraction for **causal** functions.

- ▶ Toward semi-automatic approach to synthesis.
- ▶ Approach inspired by realizability.

[Kleene (1945), ...]

Extraction from proofs

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[Kleene (1945), ...]

Analogous example: extraction for intuitionistic arithmetic (HA)

If $HA \vdash \forall x \exists y \varphi(x, y)$, there is an algorithm computing

$f : \mathbb{N} \rightarrow \mathbb{N}$ recursive such that $\forall x \varphi(x, f(x))$

- ▶ A subset of classical arithmetic (PA).
- ▶ As expressive as classical arithmetic. ($\varphi \mapsto \varphi^{\neg\neg}$)
- ▶ Can be refined to System T functions.

[Gödel (1930s)]

Analogy

Classical system	MSO(\mathbb{N})	PA
Realizers	Causal functions	System T
Intuitionistic system	???	HA

Intuitionistic version of MSO

$$\varphi, \psi ::= \alpha \mid \varphi \wedge \psi \mid \exists X \varphi \mid \neg \varphi$$

Quantification over individuals encoded as usual

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Glivenko's theorem for SMSO

MSO $\vdash \varphi$ if and only if SMSO $\vdash \neg\neg\varphi$

- Negation erases computational contents.

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Extraction of f.s. causal functions

$\text{SMSO} \vdash \exists y \neg\neg\varphi(x, y)$ iff there is a f.s. causal f s.t. $\text{MSO} \vdash \forall x \varphi(x, f(x))$

- ▶ Proofs $\varphi \vdash \psi$ interpreted as simulations between ND automata.

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Polarity restriction

A linear refinement LMSO [P., Riba (2018)]

- ▶ Polarized system with dualities.
- ▶ Requires the introduction of **linear** connectives.

Linear MSO (LMSO)

$$\varphi, \psi ::= \alpha \mid \varphi \otimes \psi \mid \varphi \wp \psi \mid \varphi \multimap \psi \mid \forall X \varphi \mid \exists X \varphi \mid !\varphi^- \mid ?\varphi^+ \mid \dots$$

A linear refinement LMSO [P., Riba (2018)]

- ▶ Polarized system with dualities.
- ▶ Requires the introduction of **linear** connectives.

Linear MSO (LMSO)

$\varphi, \psi ::= \alpha \mid \varphi \otimes \psi \mid \varphi \wp \psi \mid \varphi \multimap \psi \mid \forall X \varphi \mid \exists X \varphi \mid !\varphi^- \mid ?\varphi^+ \mid \dots$

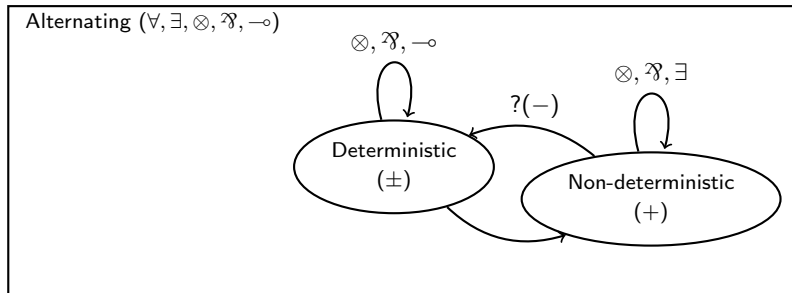
Alternating ($\forall, \exists, \otimes, \wp, \multimap$)

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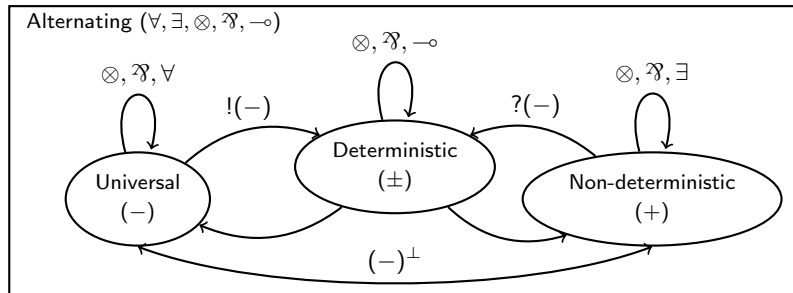
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SMSO \approx restriction to positives

Expressivity and proof extraction for LMSO

Conservativity

LMSO \rightarrow MSO

$\varphi \mapsto \lceil \varphi \rceil$

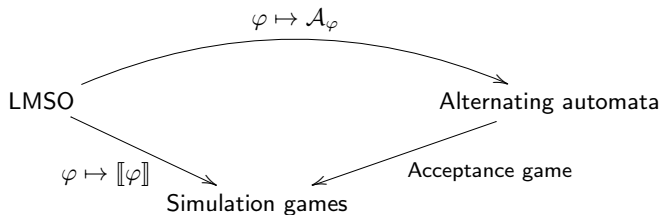
If LMSO $\vdash \varphi$, then MSO $\vdash \lceil \varphi \rceil$.

Expressivity

MSO \rightarrow LMSO

$\varphi \mapsto \varphi^L$

If MSO $\vdash \varphi$, then LMSO $\vdash \varphi^L$.



Expressivity and proof extraction for LMSO

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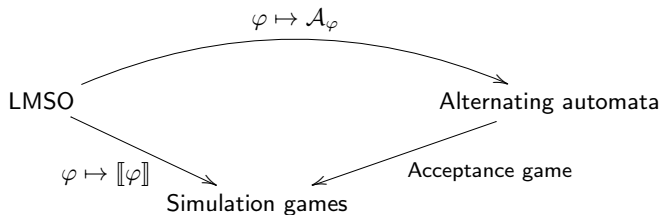
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Extraction of f.s. causal functions

$\text{LMSO} \vdash \forall x \exists y \varphi^L(x, y)$ iff there is a f.s causal f s.t. $\text{MSO} \vdash \forall x \varphi(x, f(x))$

- ▶ LMSO includes *Full Intuitionistic Multiplicative Linear Logic*.

[Hyland, de Paiva (1993)]

- ▶ Similarities with Dialectica categories DC:

[de Paiva (1989,1991)]

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- ▶ Linear *Markov principle* and *independence of premise*.

- ▶ A **classically false** choice-like scheme

$$\forall x \in \Sigma^\omega \exists y \in \Gamma^\omega \varphi(x, y) \quad \dashv\!\!\dashv \quad \exists f \in (\Sigma \rightarrow \Gamma)^\omega \forall x \in \Sigma^\omega \varphi(x, f(x))$$

$f(x)$ for pointwise application

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Double linear-negation elimination

For every φ , there is a realizer $(\varphi \multimap \perp) \multimap \perp \quad \multimap \quad \varphi$

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$f(x)$ for pointwise application

Double linear-negation elimination

For every φ , there is a realizer $(\varphi \multimap \perp) \multimap \perp \multimap \varphi$
but no canonical iso in general!

- ▶ Also holds in DC if the base satisfies choice.

The above logic can be defined without reference to automata.

- ▶ ω -word automata guarantee decidability properties. . .
- ▶ But they are not needed to extract realizers.

Why automata?

The above logic can be defined without reference to automata.

- ▶ ω -word automata guarantee decidability properties. . .
- ▶ But they are not needed to extract realizers.

↪ A purely logical reformulation of LMSO using categorical semantics.

Goals

- ▶ Purely syntactic transformations.
- ▶ Understand links with typed realizability and Dialectica.

Define the category \mathbb{M} of causal functions

- ▶ Objects: sets of streams Σ^ω for Σ finite
- ▶ Morphisms: finite-state causal functions
- ▶ Cartesian products $\Sigma^\omega \times \Gamma^\omega \simeq (\Sigma \times \Gamma)^\omega$, but **not** cartesian-closed

Finite-state causal functions as terms

Define the category \mathbb{M} of causal functions

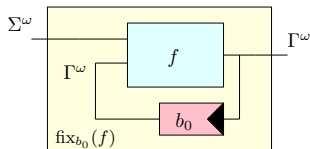
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Inductive presentation

$$\frac{f : \Sigma \rightarrow \Gamma}{f^\omega : \Sigma^\omega \rightarrow \Gamma^\omega}$$

$$\frac{f : \Sigma^\omega \times \Gamma^\omega \rightarrow \Gamma^\omega \quad b_0 \in \Gamma}{\text{fix}_{b_0}(f) : \Sigma^\omega \rightarrow \Gamma^\omega}$$

+ closure under composition



\approx guarded recursion $\text{fix} : A^{\blacktriangleright A} \rightarrow A$

topos of trees

FOM (First-Order Mealy)

$$\varphi, \psi ::= t =_{\Sigma^\omega} u \mid \varphi \wedge \psi \mid \neg \varphi \mid \exists x \in \Sigma^\omega \varphi$$

- ▶ Typed variables stand for streams, terms for every f.s. causal functions.

Proposition

FOM and MSO(\mathbb{N}) are interpretable in one another.

- ▶ Justifies focusing on FOM.

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Tarskian semantics (categorical logic)

- ▶ Regard \mathbb{M} as a multi-sorted Lawvere theory.

\rightsquigarrow Tarskian semantics \approx indexed category, from global section functor Γ

$$\Gamma : \Sigma^\omega \longmapsto \text{Hom}_{\mathbb{M}}(1^\omega, \Sigma^\omega)$$

$$\Sigma^\omega \longmapsto (\mathcal{P}(\Gamma(\Sigma^\omega)), \subseteq)$$

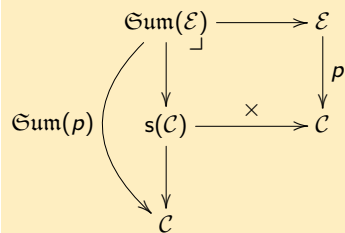
SMO and the simple fibration

Simple slice $\mathcal{C} // X =$ full subcategory of \mathcal{C} / X with objects

$$X \times Y \xrightarrow{\pi} X$$

\rightsquigarrow the simple fibration $s(\mathcal{C}) \rightarrow \mathcal{C}$

The construction $\mathfrak{S}um$



- ▶ $\mathfrak{S}um(p)$ -predicate: $(U, \varphi(a, u))$
 U object of \mathcal{C} , φ over $A \times U$ (in p)
 $\approx \exists u : U \varphi(a, u)$
- ▶ Freely adds existential quantifications
(simple sums)
- ▶ Reminiscent of typed realizability
realizers in \mathcal{C}

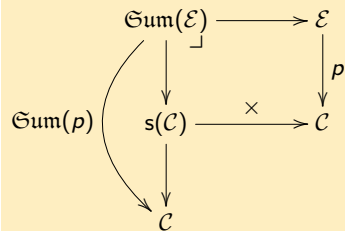
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Reconstructing SMSO

Simulations of non-deterministic automata $\approx \mathfrak{S}um$ applied to FOM

Fibered Dialectica

[Hyland (2001)]

$\mathcal{D}ial \cong \mathcal{S}um \circ \mathcal{P}rod$

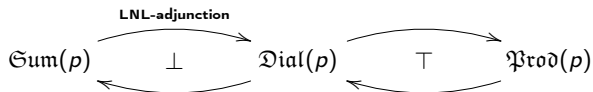
$\mathcal{P}rod(p) \cong \mathcal{S}um(p^{op})^{op}$

[Hofstra (2011)]

- ▶ $\mathcal{D}ial(p)$ -predicate over $A \approx (U, X, \varphi(a, u, x))$

think $\exists u \forall x \varphi(a, u, x)$

- ▶ interprets full intuitionistic MLL+FO



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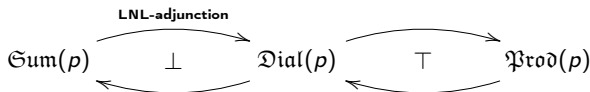
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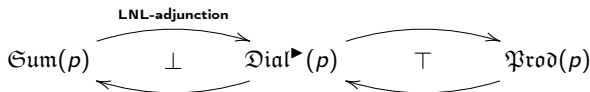
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Realized Dialectica-like construction $\mathcal{D}ial^{\blacktriangleright}$

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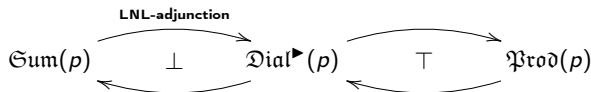
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- ▶ Only over a CCC extension of \mathbb{M}

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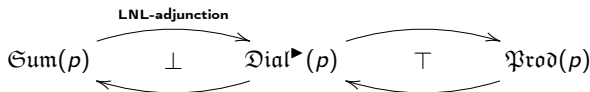
$$\mathfrak{P}\text{rod}(p) \cong \text{Sum}(p^{\text{op}})^{\text{op}} \quad [\text{Hofstra (2011)}]$$

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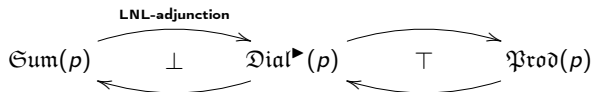
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- ▶ Polarity restrictions \approx model of LMSO

(restricted exponentials)

Monadic Second-Order logic

Part I: Reverse Mathematics

Part II: proof systems for Church's synthesis

Conclusion

Summary

Axiomatic strength of two classical MSO theories.

- ▶ In the context of Reverse Mathematics.
- ▶ Strong link between Σ_2^0 -induction and $\text{MSO}(\mathbb{N})$.
- ▶ Preliminary results on $\text{MSO}(\mathbb{Q})$.

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Related work

- ▶ Characterizations of the topological complexity of MSO-definable sets.
- ▶ Extension to the Reverse-mathematical analysis to intuitionistic logic.
[Lichter and Smolka (2018)]
- ▶ Conservativity results for cyclic arithmetic.
[Simpson (2017), Das (2019)]

Summary

- ▶ Realizability models based on simulations between automata
- ▶ Abstract reformulation link with Dialectica and typed realizability
- ▶ Complete extension of LMSO omitted from the talk [P., Riba (2019)]

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Related work

- ▶ Fibrations of tree automata [Riba (2015)]
- ▶ Good-for-games automata [Henziger, Piterman (2006), Kuperberg Skrzypczak (2015)]

Some further questions

- ▶ Realizability for *continuous* functions $\Sigma^\omega \rightarrow \Gamma^\omega$?
- ▶ Extensions of $\mathcal{D}ial$ for fibrations over the topos of trees?
 $\mathfrak{F}am(\mathfrak{F}am(p^{op})^{op})$ instead of $\mathcal{D}ial(p)$
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Thanks for your attention! Questions?

RCA_0 is defined by restricting *induction* and *comprehension*

Comprehension axiom

For every formula $\phi(n)$ (with $X \notin \text{FV}(\phi)$)

$$\exists X \forall n \in \mathbb{N} (\phi(n) \Leftrightarrow n \in X)$$

- ▶ RCA_0 : restricted to Δ_1^0 formulas

recursive comprehension

Induction axiom

To prove that $\forall n \in \mathbb{N} \phi(n)$ it suffices to show

- ▶ $\phi(0)$ holds
- ▶ for every $n \in \mathbb{N}$, $\phi(n)$ implies $\phi(n+1)$

- ▶ RCA_0 : restricted to Σ_1^0 formulas.

$\exists n \delta(n)$ with $\delta \in \Delta_1^0$

- ▶ Equivalent to minimization principles and comprehension for finite sets.

Additive Ramsey

Let M be a monoid. For every map $f : [\mathbb{N}]^2 \rightarrow M$ such that $f(i, j)f(j, k) = f(i, k)$, there exists an infinite set $X \subseteq \mathbb{N}$ and $c \in M$ such that $f(i, j) = c$ for $i, j \in X$.

Theorem

Over RCA_0 , additive Ramsey is equivalent to $\Sigma_2^0\text{-IND}$.

Combinatorics for coloring over \mathbb{Q}

Let D be a dense linear order ($\simeq \mathbb{Q}$).

A function $f : D \rightarrow X$ is called *homogeneous* if $f^{-1}(x)$ is either dense or empty for every $x \in X$.

The shuffle principle

For any coloring $c : \mathbb{Q} \rightarrow \llbracket 0, n \rrbracket$, there is $]x, y[$ such that $c|_{]x, y[}$ is *homogeneous*.

- ▶ the key additional principle behind the usual inductive argument in [Carton, Colcombet, Puppis (2015)]

Shelah's additive Ramseyan theorem

Let M be a monoid. For every map $f : [\mathbb{Q}]^2 \rightarrow M$ such that $f(q, r)f(r, s) = f(q, s)$, there exists an interval $I \subseteq \mathbb{Q}$ and a finite partition into finitely many dense sets D_i of I such that f is constant over each $[D_i]^2$.

- ▶ the key additional principle behind the usual inductive argument in [Shelah (1975)]

The Büchi-Landweber theorem

Consider a formula $\varphi(u, x)$.

\rightsquigarrow Infinite 2-player game \mathcal{G}_φ between **P** and **O**.

O	x_0	x_1	\dots	x_n	\dots
P	u_0	u_1	\dots	u_n	\dots

$(u \in U^\omega, x \in X^\omega)$

P wins
 \iff
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P-strategies $\simeq X^+ \rightarrow U$
causal functions

O-strategies $\simeq U^* \rightarrow X$
eager causal functions

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Theorem [Büchi-Landweber (1969)]

Suppose φ is MSO-definable. The game \mathcal{G}_φ is determined:

- ▶ Either there exists a finite-state **P**-strategy $s_P(x)$ s.t. $\forall x \in X^\omega \varphi(s_P(x), x)$
- ▶ Or there exists a finite-state **O**-strategy $s_O(u)$ s.t. $\forall u \in U^\omega \neg\varphi(u, s_O(u))$

Uniform non-deterministic automata

Tuples $\mathcal{A} = (Q, q_0, U, \delta_{\mathcal{A}}, \Omega_{\mathcal{A}}) : \Sigma$ where

- ▶ U a set of *moves* \simeq amount of non-determinism
 - ▶ transition function $\delta_{\mathcal{A}} : \Sigma \times Q \times U \rightarrow Q$ induces $\delta_{\mathcal{A}}^* : \Sigma^\omega \times U^\omega \rightarrow Q^\omega$
 - ▶ $\Omega_{\mathcal{A}} \subseteq Q^\omega$ reasonable acceptance condition (parity, Muller, ...)
-
- ▶ Same definable languages $\mathcal{L}(\mathcal{A}) = \{w \mid \exists u \delta_{\mathcal{A}}^*(w, u)\}$ $U \simeq Q$

The realizability notion for SMSO

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Simulations $\mathcal{A} \Vdash f : \mathcal{B}$

Finite-state causal function $f : \Sigma^\omega \times U^\omega \rightarrow V^\omega$ such that

$$\forall w \in \Sigma^\omega \forall u \in U^\omega \quad \delta_{\mathcal{A}}^*(w, u) \in \Omega_{\mathcal{A}} \Rightarrow \delta_{\mathcal{A}}^*(w, f(w, u)) \in \Omega_{\mathcal{B}}$$

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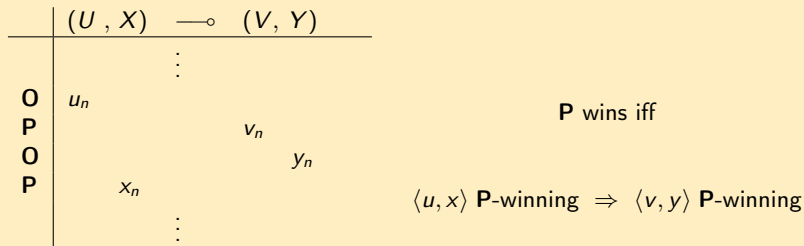
- ▶ If $\mathcal{A} \Vdash \mathcal{B}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$
- ▶ Natural interpretation for \exists , \wedge and \neg for deterministic automata...

Alternating uniform automata

Define a notion of *alternating* uniform automata $(Q, q_0, U, X, \delta, \Omega)$

- ▶ sets of **P**-moves U and **O**-moves X
- ▶ $\delta : \Sigma \times Q \times U \times X \rightarrow Q$
- ▶ $w \in \mathcal{L}(\mathcal{A})$ iff **P** wins an *acceptance game*

Simulation game



- ▶ $X \simeq 1 \rightsquigarrow$ non-deterministic uniform automata
- ▶ $U \simeq X \simeq 1 \rightsquigarrow$ deterministic automata

trivial simulations