Implicit automata in typed λ-calculi

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regexps, automata (deterministic or not), MSO, ...

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using high-level programming languages

Big project: the same thing for automata instead of complexity

programming paradigm	declarative	functional
complexity classes	descriptive complexity	implicit complexity
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Typed λ-calculi ($\cong$ constructive logics): theoretical basis for e.g. Haskell

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Other "implicit automata" results: cf. MIMUW automata seminar 2020-05-13
DeYoung & Pfenning 2016 / Kuperberg et al. 2019, using *circular proofs*

The λ-calculus

λ-terms: meant to be a naive syntactic theory of functions

$$t, u ::= x \mid t\ u \mid \lambda x.\ t \qquad\qquad f\ t \approx f(t) \qquad \lambda x.\ t \approx (x \mapsto t)$$

Notational conventions: $t\ u\ v = (t\ u)\ v$ and $\lambda x.\ t\ u = \lambda x.\ (t\ u)$

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For $f : x \mapsto x^2 + 1$, we have $f(42) = 42^2 + 1$ ...

Operational semantics: program execution by rewriting

$$(\lambda x.\ t)\ u \longrightarrow_\beta t\{x := u\}\ (t \text{ with } x \textit{ substituted} \text{ by } u)$$

We write $=_\beta$ for the smallest equivalence relation containing $\longrightarrow_\beta$

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No primitive data; encodings using functions

Useful example: booleans

$\texttt{true} = \lambda x.\ \lambda y.\ x \qquad \texttt{false} = \lambda x.\ \lambda y.\ y$

(multiple arguments: $x \mapsto (y \mapsto x) \cong (x, y) \mapsto x$)

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Also encodings of natural numbers, strings... (later!)

Untyped λ-calculus is Turing-complete

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Simple types: specifications for λ-calculus programs

$$A, B ::= o \text{ (base type)} \mid A \to B \text{ (functions from } A \text{ to } B)$$

Convention: $A \to B \to C = A \to (B \to C)$

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$$\overline{abb} = \lambda f_a.\, \lambda f_b.\, \lambda x.\, f_a\ (f_b\ (f_b\ x)) \ :\ \texttt{Str}_{\{a,b\}} = (o \to o) \to (o \to o) \to o \to o$$

Idea: $\overline{w} \approx (f_a, f_b) \mapsto f_{w[1]} \circ \ldots \circ f_{w[n]}$ for $w \in \{a, b\}^*$

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Languages: $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Bool}$ (A is any simple type)
(for $w \in \Sigma^*$, $\overline{w} : \mathsf{Str}_\Sigma$ therefore $\overline{w} : \mathsf{Str}_\Sigma\{o := A\}$)

Regular languages in the simply typed λ-calculus

Theorem (Hillebrand & Kanellakis 1996)

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Proof ideas for non-trivial direction ($\Longleftarrow$):

- $w \in \Sigma^* \mapsto \overline{w} : \{t \mid t : \mathsf{Str}_\Sigma\{o := A\}\}/=_\beta$ is a *monoid morphism*

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Canonical semantics: realizes the naive interpretation

- for *types*: $[\![o]\!] =$ some 2-element set, $[\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$
- for *terms*: $t : A \rightsquigarrow [\![t]\!] \in [\![A]\!]$ inductively defined, e.g. $[\![f\ x]\!] = [\![f]\!]\,([\![x]\!])$
- soundness: $t =_\beta u \implies [\![t]\!] = [\![u]\!]$
- preserves some relevant information: $[\![\mathtt{true}]\!] \neq [\![\mathtt{false}]\!]$

String-to-string functions

Open problem (that we don't solve :-( )

Nice characterization (preferably automata-theoretic) of
$\mathsf{Str}_\Gamma\{o := A\} \to \mathsf{Str}_\Sigma$ in the simply typed λ-calculus?

($\mathsf{Str}_\Gamma \to \mathsf{Str}_\Sigma$ described by base functions + composition [Zaionc 1987])

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Next: more restrictive type systems $\rightsquigarrow$ smaller classes of transductions
+ one smaller class of languages

actual results instead of open problems

Reminder: star-free languages and aperiodic monoids

Our goal now: characterize *star-free languages* (ICALP 2020 paper).

Definition

A language is *star-free* if is defined by some regexp *without repetition star* L^*, but potentially with *complementation* $L^c = \Sigma^* \setminus L$.

Example with $\Sigma = \{a, b\}$: $(ab)^* = (b\varnothing^c | \varnothing^c a | \varnothing^c aa\varnothing^c | \varnothing^c bb\varnothing^c)^c$

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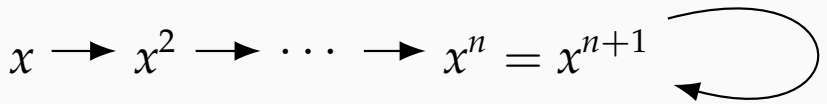

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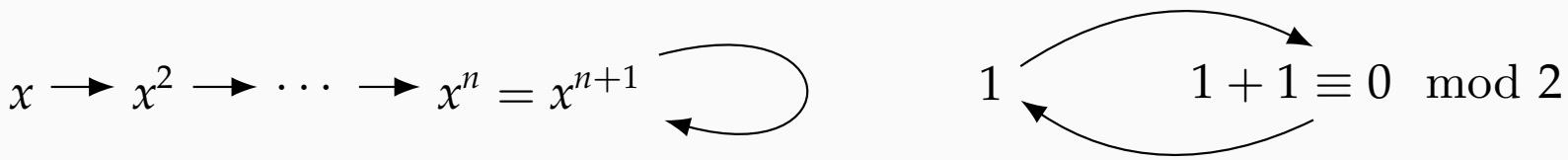

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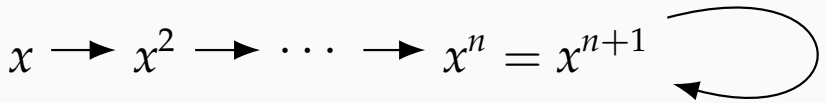

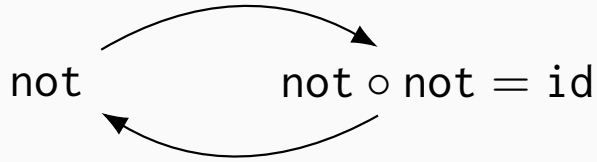

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Idea 2: affine typing

Prohibit duplication: "a function should use its arguments *at most once*"

Variants of *linear logic* are widely used in implicit complexity!

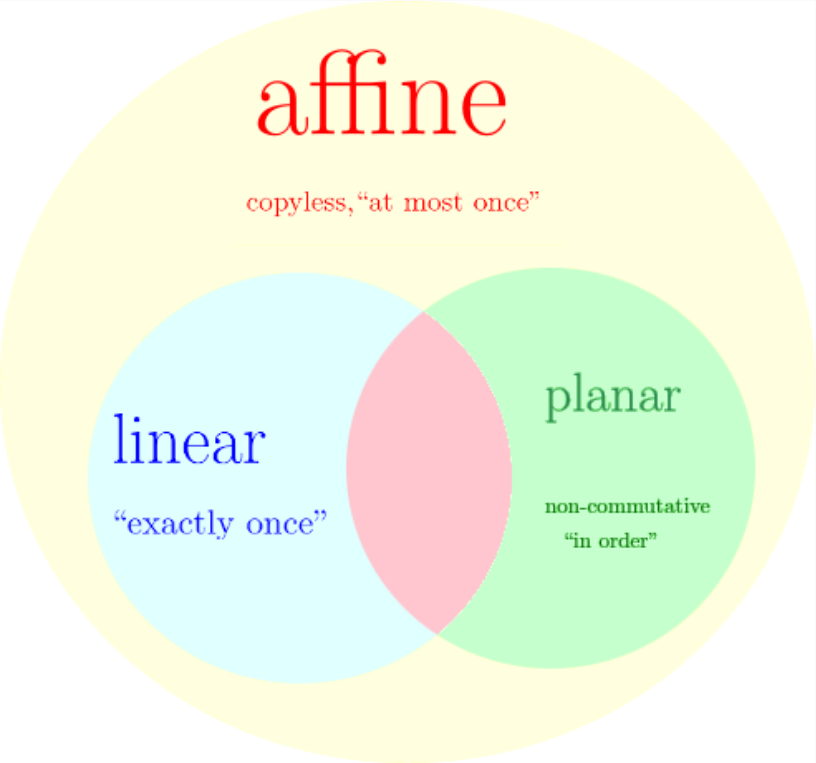
linear = "exactly once"; would work for star-free languages, but not for string functions

Substructural arrow types

$A \multimap B$: a restricted arrow type

$$A, B ::= o \mid A \to B \mid A \multimap B$$

$$\frac{\Delta \vdash t : A \multimap B \qquad \Delta' \vdash u : A}{\Delta, \Delta' \vdash t\ u : B}$$



	linear	affine	planar
$\lambda f\ x.\ f\ x\ x$	✗	✗	✗
$\lambda x\ y.\ x\ y$	✓	✓	✓
$\lambda x\ y.\ y$	✗	✓	✓
$\lambda x\ y.\ y\ x$	✓	✓	✗

Planar string languages

Church encoding with linear/affine types [Girard 1987]

$\overline{abb} = \lambda^{\rightarrow} f_a.\ \lambda^{\rightarrow} f_b.\ \lambda^{\circ} x.\ f_a\ (f_b\ (f_b\ x)) : \mathsf{Str}_{\{a,b\}} = (o \multimap o) \to (o \multimap o) \to o \multimap o$

- Take $\mathsf{Bool} = o \multimap o \multimap o$ in the affine setting.

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Theorem (Nguyễn & P., ICALP 2020)

$L \subseteq \Sigma^*$ *is* star-free $\iff$ L *is defined by some* $t : \mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Bool}$ *in our* non-commutative affine λ-calculus *with* A purely affine*, i.e. containing no* '$\to$'

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With commutative affine types: regular languages instead

A characterization of star-free languages: proof ideas

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Key lemma for ($\Longrightarrow$)

aperiodic sequential functions $\subseteq \mathsf{Str}_\Gamma\{o := A\} \multimap \mathsf{Str}_\Sigma$ with purely affine A

Proved using the Krohn–Rhodes decomposition… (linear case: more tricky)

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Key lemma for ($\Longleftarrow$)

For purely affine A, the non-commutative λ-terms of type $A \multimap A$ (modulo $=_\beta$) form a finite and aperiodic monoid.

The paper has a brute force proof; more conceptual (found afterwards):

- compile into "planar diagrams" (variant of Kauffman monoids)
  $\exists$ well-known connections: non-commutativity in linear logic $\leftrightarrow$ planarity
- characterize $\leq_{\mathcal{H}}$ on planar diagrams, deduce $\mathcal{H}$-triviality

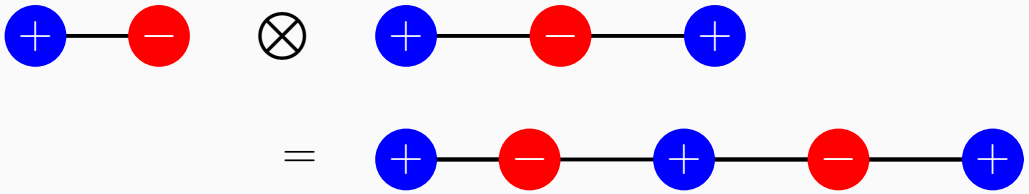
Planar diagrams (1/3): interpreting types

Interpretation of types

$[\![(o \multimap o) \multimap o]\!]$ $=$ 

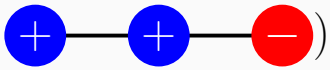
Tensorial product

put side-by-side



Duality

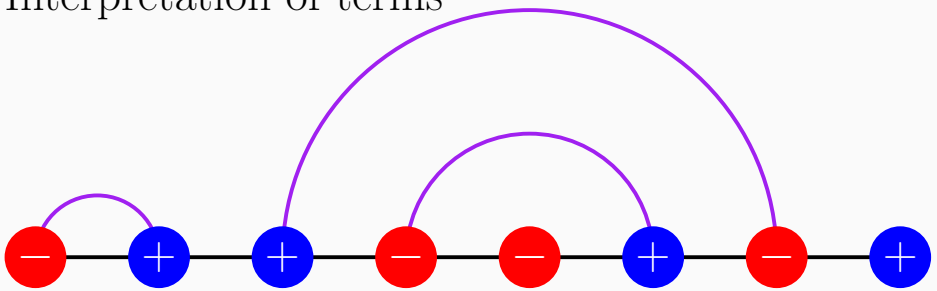
reverse the order and switch polarity

$($  $)^*$ $=$ 

$A \multimap B = A^* \otimes B$

Planar diagrams (2/3): interpreting terms and composition

Interpretation of terms



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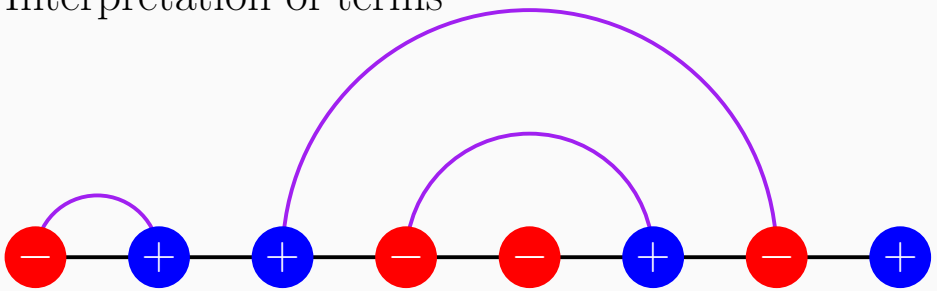
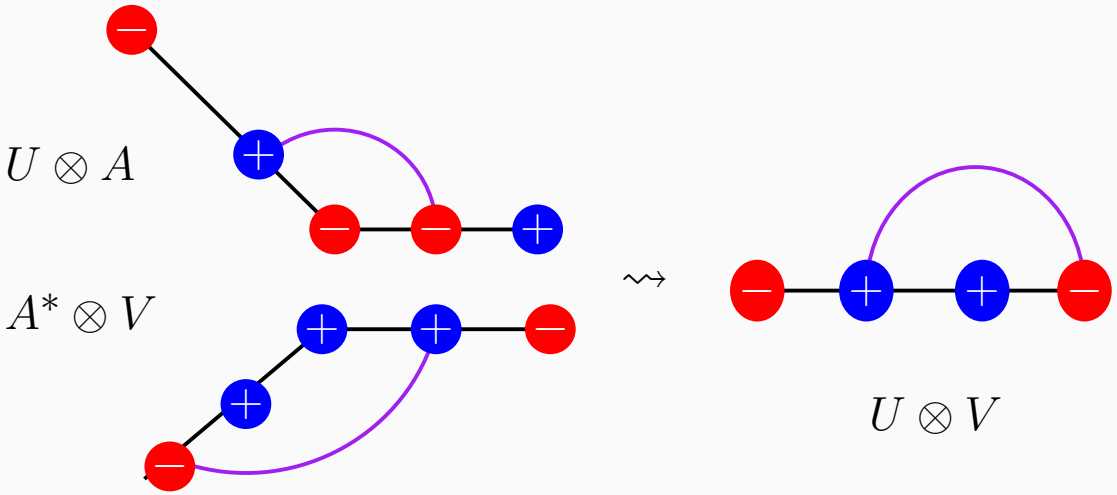
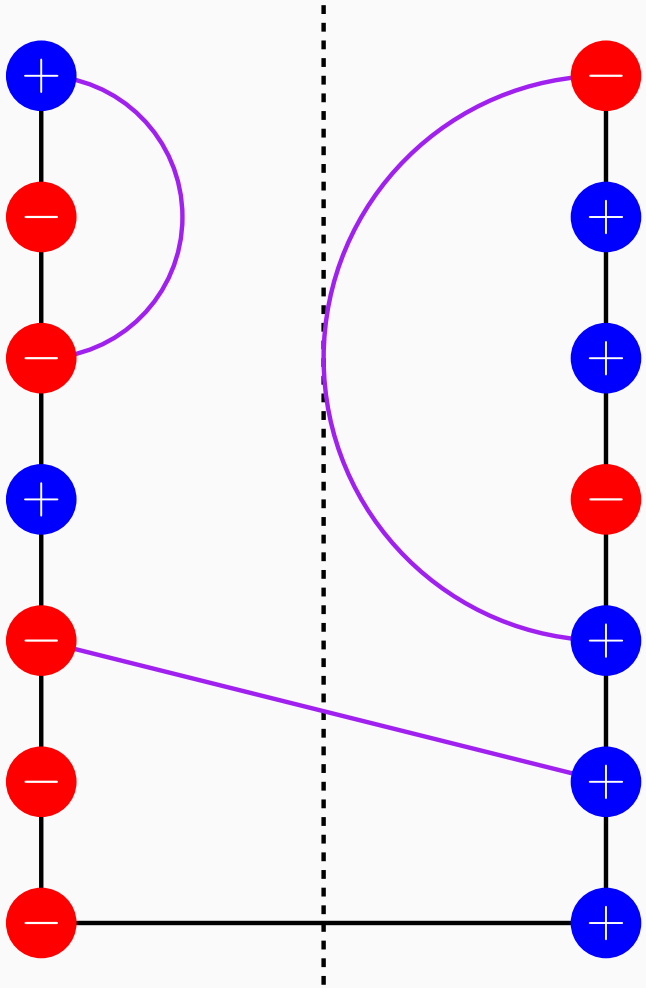


Composition



Planar diagrams (3/3): aperiodicity

Monoidal closed cat $\Rightarrow$ it suffices to show that all monoids $A \multimap A$ are aperiodic



$A^* \otimes A$

$\mathcal{R}$-class $\approx$

internal wiring on the left

$+$

positions of the input nodes

$\mathcal{L}$-class $\approx$

internal wiring on the right

$+$

position of the output nodes

planar $\Rightarrow$ $\mathcal{H}$-trivial

Back to string-to-string functions

Key lemma for star-free $\Longrightarrow$ non-commutatively λ-definable

aperiodic sequential functions $\subseteq \mathsf{Str}_\Gamma\{o := A\} \multimap \mathsf{Str}_\Sigma$ with purely affine A

Does $(\supseteq)$ hold?

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Does ($\supseteq$) hold? **No:** sequential fn $\subsetneq$ rational fn $\subsetneq$ **regular fn**, and

Theorem

In the commutative (*resp. non-commutative*) *affine λ-calculus,*
$\mathsf{Str}_\Gamma\{o := A\} \multimap \mathsf{Str}_\Sigma$ *with purely affine A corresponds to the* regular (*resp.* aperiodic regular) *functions.*

(also called respectively MSO transductions and FO transductions)

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One possible definition for regular functions:
copyless *streaming string transducers* [Alur & Černý 2010]
$\rightsquigarrow$ sounds suspiciously like affine types!

Streaming string transducers

- Finite set of Σ^*-valued *registers* e.g. $R = \{X, Y\}$
- Initial values $R \to \Sigma^*$ e.g. $X_{\text{init}} = Y_{\text{init}} = \varepsilon$
- *Register update function* e.g. $a \mapsto \begin{cases} X := Xa \\ Y := aY \end{cases} \qquad b \mapsto \begin{cases} X := Xb \\ Y := bY \end{cases}$
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Execution over *abaa*: start with

$$X = \varepsilon \qquad Y = \varepsilon$$

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Execution over *abaa*:

$$X = ab \qquad Y = ba$$

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Execution over *abaa*:

$$X = aba \qquad Y = aba$$

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Copyless SST: each register appears *at most once* on RHS of $:=$
(for each fixed input letter, at most once among all the associated $:=$)

Categorical automata theory meets denotational semantics (1)

A framework for "single-pass" automata [Colcombet & Petrişan 2017]

- internal memory = object of a *category*
- transitions = morphisms (and [letter $\mapsto$ transition] = functor)

- DFA = automata over the category of finite sets
- Copyless SSTs $\approx$ start from a category $\mathcal{R}$ of copyless register updates
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Proof strategy for affinely λ-definable $\Longrightarrow$ regular function

Define a *functor* $\mathcal{L} \to \mathcal{R}_\oplus$ preserving enough structure

Useful fact: there is a canonical functor from $\mathcal{L}$ to any *affine symmetric monoidal closed category* (i.e. categorical semantics of the purely affine λ-calculus)

Unfortunately $R_\oplus$ is not monoidal closed...

Categorical automata theory meets denotational semantics (2)

- $\mathcal{L}$: category of purely affine λ-terms (w/ const $f_c : o \multimap o$ for $c \in \Sigma$)
- $\mathcal{R}$: category of finite sets of registers and copyless assignments
- $\mathcal{R}_\oplus$: free finite coproduct completion of the latter (add states)

We consider the free finite *product* completion: $\mathcal{C}_\& = ((\mathcal{C}^{\mathrm{op}})_\oplus)^{\mathrm{op}}$

Theorem

$(\mathcal{R}_\&)_\oplus$ *is symmetric monoidal closed* (*and almost affine*).

$\Longrightarrow$ $(\mathcal{R}_\&)_\oplus$-automata are at least as powerful as $\mathcal{L}$-automata!

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- logic POV: connection with Dialectica interpretation / DC categories
  (coproducts of products $\approx \exists u.\, \forall x.\, \varphi(u, x)$)

Summary

λ-calculus	automata	status
STLC, $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Bool}$	regular languages	✓[Hillebrand,Kanellakis 96]
affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Bool}$	regular languages	✓[Nguyễn, P. 20]
affine planar, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Bool}$	star-free languages	✓[Nguyễn, P. 20]
linear, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathit{NC}\mathsf{Bool}$	regular languages	✓
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λ-calculus	transducers	status
STLC, $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Str}_\Gamma$	variant of CPDA???	???
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planar affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	FO regular	✓?
linear with additives, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	regular functions	✓(coming soon)
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affine, $\mathsf{Str}_\Sigma \multimap o \to o \to o$	flip-flop monoids	✓
STLC, $\mathsf{Str}_\Sigma \to o \to o \to o$	$BC(\Sigma_1)$	??

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STLC, $\mathsf{Str}_\Sigma \to \mathsf{Str}_\Gamma$	"extended polynomials"	[Zaionc 1987]

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- A more principled description of the variations?
- What about trees ?

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affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	regular	✓?
planar affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	FO regular	✓?
linear with additives, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	regular functions	✓(coming soon)
linear with additives, $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Str}_\Gamma$	comparison-free polyregular	??
STLC, $\mathsf{Str}_\Sigma \to \mathsf{Str}_\Gamma$	"extended polynomials"	[Zaionc 1987]

- A more principled description of the variations?
- What about trees ?

Summary

λ-calculus	automata	status
STLC, $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Bool}$	regular languages	✓[Hillebrand,Kanellakis 96]
affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Bool}$	regular languages	✓[Nguyễn, P. 20]
affine planar, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Bool}$	star-free languages	✓[Nguyễn, P. 20]
linear, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathit{NC}\mathsf{Bool}$	regular languages	✓
linear planar, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathit{NC}\mathsf{Bool}$	star-free languages	✓
affine, $\mathsf{Str}_\Sigma \multimap o \to o \to o$	flip-flop monoids	✓
STLC, $\mathsf{Str}_\Sigma \to o \to o \to o$	$BC(\Sigma_1)$	??

λ-calculus	transducers	status
STLC, $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Str}_\Gamma$	variant of CPDA???	???
affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	regular	✓?
planar affine, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	FO regular	✓?
linear with additives, $\mathsf{Str}_\Sigma\{o := A\} \multimap \mathsf{Str}_\Gamma$	regular functions	✓(coming soon)
linear with additives, $\mathsf{Str}_\Sigma\{o := A\} \to \mathsf{Str}_\Gamma$	comparison-free polyregular	??
STLC, $\mathsf{Str}_\Sigma \to \mathsf{Str}_\Gamma$	"extended polynomials"	[Zaionc 1987]

- A more principled description of the variations?
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Thanks for your attention! Questions?